Example 8.13: *Gradient estimation of a constant mass*

Consider the problem of mass estimation for the dynamics

$$u = m\,w(t) + d(t)$$

with $w(t) = \sin(t)$, and $d(t)$ is interpreted as either disturbance or measurement noise. The true mass is assumed to be $m = 2$. When the disturbance $d(t) = 0$, the estimation results are shown in the left plot in Figure 8.21. It is seen that larger gain corresponds to faster convergence, as expected. When the disturbance is $d(t) = 0.5\sin(20t)$, the estimation results are shown on the right plot in Figure 8.21. It is seen that larger estimation gain leads to larger estimation error.

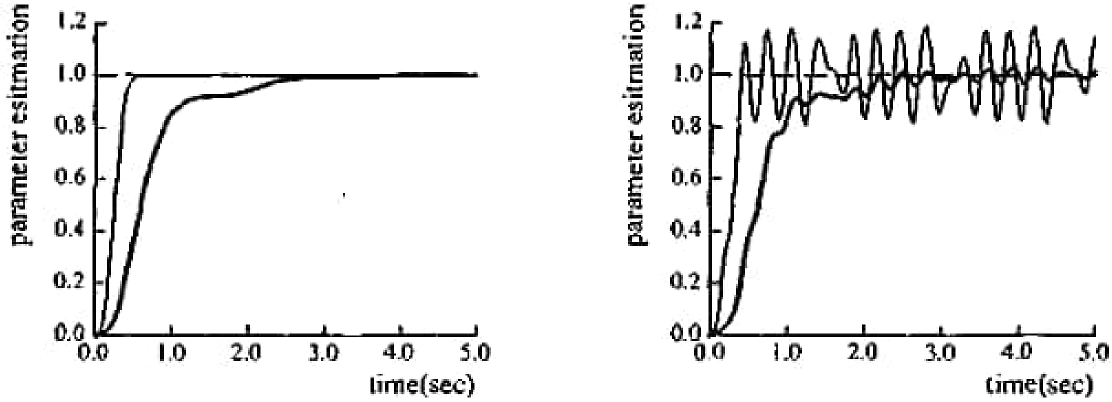

Figure 8.21 : gradient method, left: without noise, right: with noise

The following simple simulation illustrates the behavior of the gradient estimator in the presence of both parameter variation and measurement noise.

Example 8.3: A first-order plant

Consider the control of the unstable plant

$$\dot{y} = y + 3u$$

using the previously designed adaptive controller. The plant parameters $a_p = -1, b_p = 3$ are assumed to be unknown to the adaptive controller. The reference model is chosen to be

$$\dot{x}_m = -4x_m + 4r$$

i.e., $a_m = 4, b_m = 4$. The adaptation gain γ is chosen to be equal to 2. The initial values of both parameters of the controller are chosen to be 0, indicating no *a priori* knowledge. The initial conditions of the plant and the model are both zero.

Example 8.4: simulation of a first-order nonlinear plant

Assume that a nonlinear plant is described by the equation

$$\dot{y} = y + y^2 + bu \tag{8.35}$$

This differs from the unstable plant in Example 8.3 in that a quadratic term is introduced in the plant dynamics.

Let us use the same reference model, initial parameters, and design parameters as in Example 8.3. For the reference signal $r(t) = 4$, the results are shown in Figure 8.11. It is seen that the tracking error converges to zero, but the parameter errors are only bounded. For the reference signal $r(t) = 4\sin(3t)$, the results are shown in Figure 8.12. It is noted that the tracking error and the parameter errors for the three parameters all converge to zero. □

In this example, it is interesting to note two points. The first point is that a single sinusoidal component in $r(t)$ allows three parameters to be estimated. The second point is that the various signals (including $\hat{a}$ and y) in this system are much more oscillatory than those in Example 8.3. Let us understand why. The basic reason is provided by the observation that nonlinearity usually generates more frequencies, and thus $\mathbf{v}(t)$ may contain more sinusoids than $r(t)$. Specifically, in the above example, with $r(t) = 4\sin(3t)$, the signal vector $\mathbf{v}$ converges to

$$\mathbf{v}(t) = [r(t) \quad y_{ss}(t) \quad f_{ss}(t)]$$

where $y_{ss}(t)$ is the steady-state response and $f_{ss}(t)$ the corresponding function value,

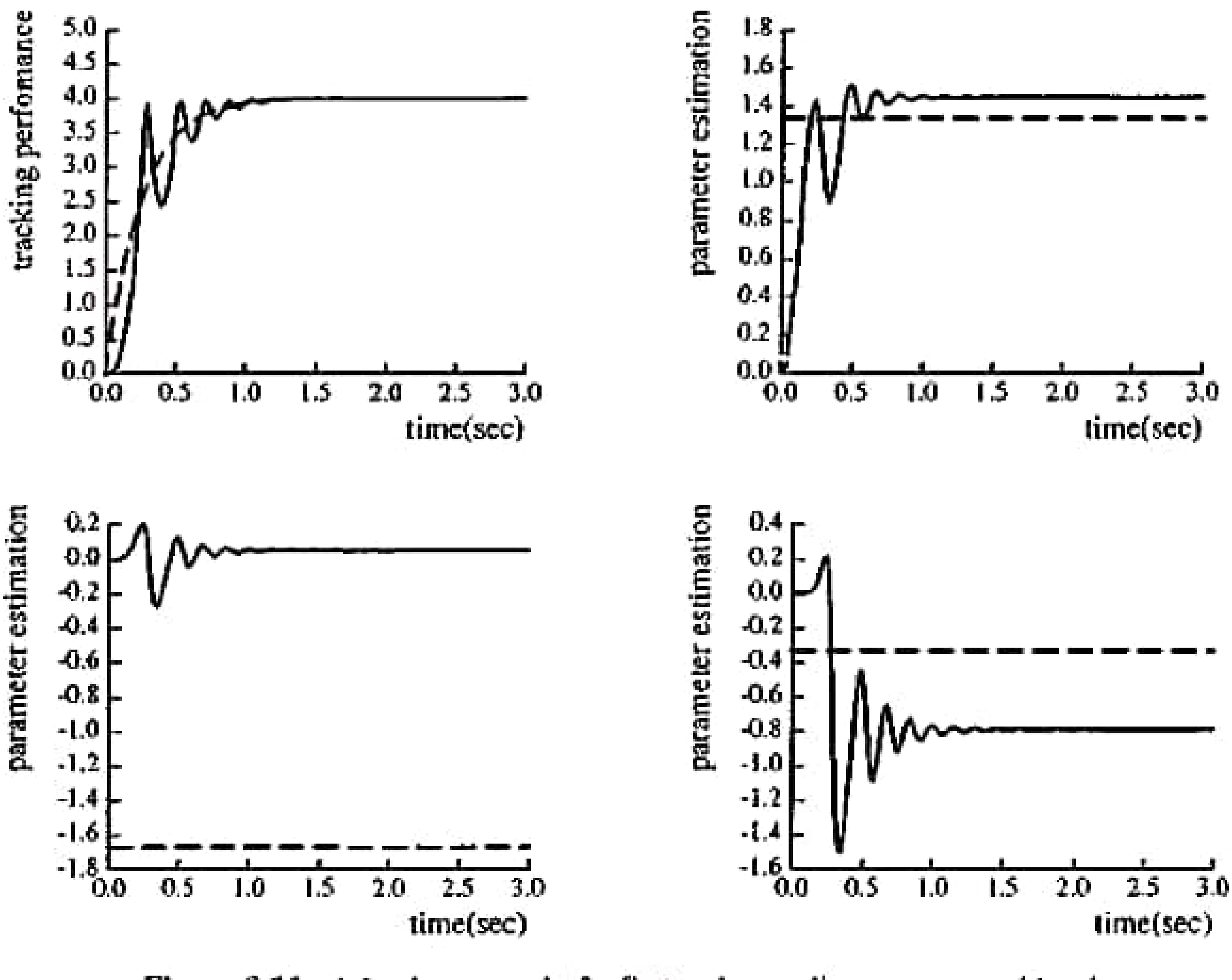

Figure 8.11 : Adaptive control of a first-order nonlinear system, $r(t) = 4$
upper left: tracking performance
upper right: parameter $\hat{a}_r$; lower left: parameter $\hat{a}_y$; lower right: parameter $\hat{a}_f$

$$y_{ss}(t) = y_m(t) = 4A\sin(3t + \phi)$$

$$f_{ss}(t) = y_{ss}^{\,2} = 16A^2\sin^2(3t + \phi) = 8\,A^2(1 - \cos(6t + 2\phi))$$

where A and ϕ are the magnitude and phase shift of the reference model at $\omega = 3$. Thus, the signal vector $\mathbf{v}(t)$ contains *two* sinusoids, with $f(y)$ containing a sinusoid at twice the original frequency. Intuitively, this component at double frequency is the reason for the convergent estimation of the *three* parameters and the more oscillatory behavior of the estimated parameters.

Example 8.5: A controller for perfect tracking

Consider the plant described by

$$y = \frac{k_p(p + b_p)}{p^2 + a_{p1}p + a_{p2}}\, u \tag{8.45}$$

and the reference model

$$y_m = \frac{k_m(p + b_m)}{p^2 + a_{m1}p + a_{m2}}\, r \tag{8.46}$$

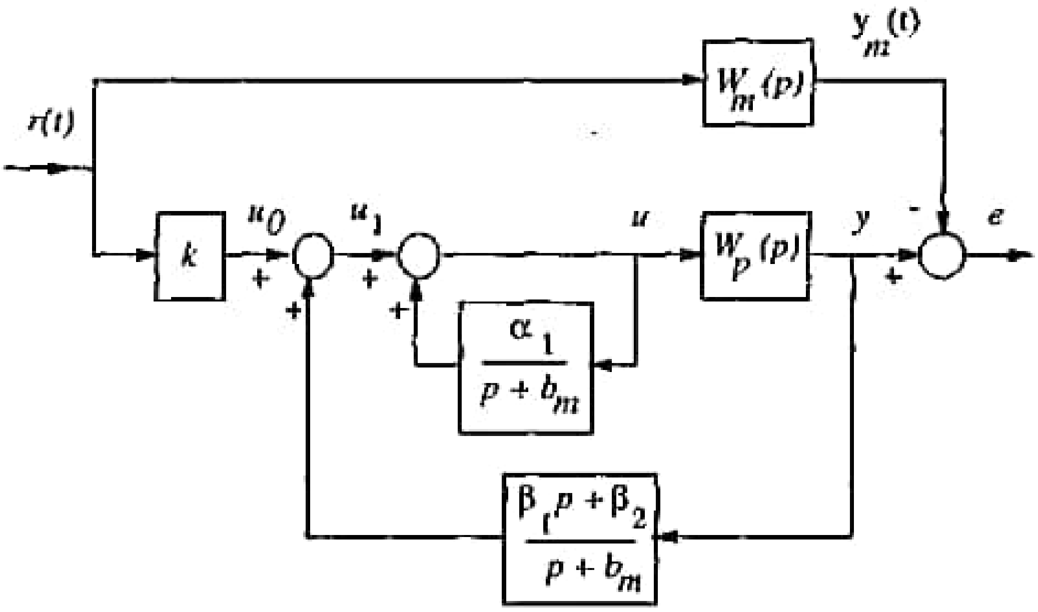

Figure 8.13 : A model-reference control system for relative degree 1

Let the controller be chosen as shown in Figure 8.13, with the control law being

$$u = \alpha_1 z + \frac{\beta_1 p + \beta_2}{p + b_m} y + k\, r \tag{8.47}$$

where $z = u/(p + b_m)$, *i.e.*, z is the output of a first-order filter with input u, and α_1, β_1, β_2, k are controller parameters. If we take these parameters to be

$$\alpha_1 = b_p - b_m$$

$$\beta_1 = \frac{a_{m1} - a_{p1}}{k_p}$$

$$\beta_2 = \frac{a_{m2} - a_{p2}}{k_p}$$

$$k = \frac{k_m}{k_p}$$

one can straightforwardly show that the transfer function from the reference input r to the plant output y is

$$W_{ry} = \frac{k_m(p + b_m)}{p^2 + a_{m1}p + a_{m2}} = W_m(p)$$

Therefore, perfect tracking is achieved with this control law, *i.e.*, $y(t) = y_m(t)$, $\forall t \geq 0$.

It is interesting to see why the closed-loop transfer function can become exactly the same as that of the reference model. To do this, we note that the control input in (8.47) is composed of three parts. The first part in effect replaces the plant zero by the reference model zero, since the transfer function from u_1 to y (see Figure 8.13) is

$$W_{u1,y} = \frac{p + b_m}{p + b_p}\;\frac{k_p(p+b_p)}{p^2+a_{p1}p+a_{p2}} = \frac{k_p(p + b_m)}{p^2 + a_{p1}p + a_{p2}}$$

The second part places the closed-loop poles at the locations of those of the reference model. This is seen by noting that the transfer function from u_o to y is (Figure 8.13)

$$W_{u0,y} = \frac{W_{u1,y}}{1 + W_f W_{u1,y}} = \frac{k_p(p + b_m)}{p^2 + (a_{p1} + \beta_1 k_p)p + (a_{p2}+\beta_2 k_p)}$$

The third part of the control law $(k_m/k_p)r$ obviously replaces k_p, the high frequency gain of the plant, by k_m. As a result of the above three parts, the closed-loop system has the desired transfer function. □

The controller structure shown in Figure 8.13 for second-order plants can be extended to any plant with relative degree one. The resulting structure of the control system is shown in Figure 8.14, where k^*, θ_1^*, θ_2^* and θ_o^* represents controller parameters which lead to perfect tracking when the plant parameters are known.

The structure of this control system can be described as follows. The block for generating the filter signal ω_1 represents an $(n-1)^{th}$ order dynamics, which can be described by

$$\dot{\boldsymbol{\omega}}_1 = \boldsymbol{\Lambda}\boldsymbol{\omega}_1 + \mathbf{h}u$$

where $\boldsymbol{\omega}_1$ is an $(n-1)\times 1$ state vector, $\boldsymbol{\Lambda}$ is an $(n-1)\times(n-1)$ matrix, and $\mathbf{h}$ is a constant vector such that $(\boldsymbol{\Lambda}, \mathbf{h})$ is controllable. The poles of the matrix $\boldsymbol{\Lambda}$ are chosen to be the

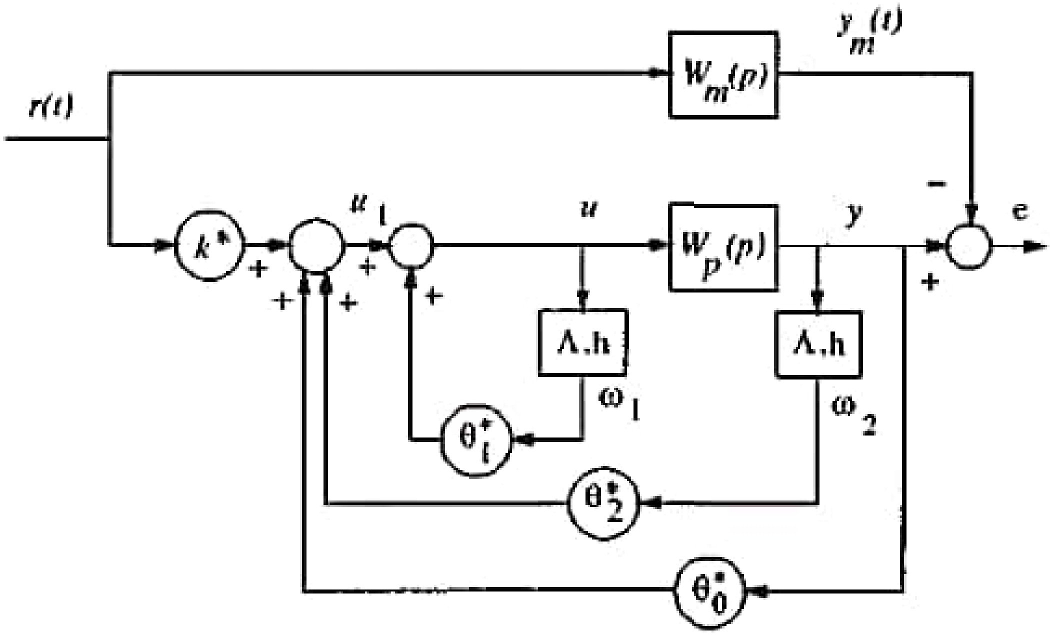

Figure 8.14 : A control system with perfect tracking

same as the roots of the polynomial $Z_m(p)$, *i.e.*,

$$\det[p\mathbf{I} - \mathbf{\Lambda}] = Z_m(p) \tag{8.48}$$

The block for generating the $(n-1)\times1$ vector ω_2 has the same dynamics but with y as input, *i.e.*,

$$\dot{\omega}_2 = \mathbf{\Lambda}\,\omega_2 + \mathbf{h}\,y$$

It is straightforward to discuss the controller parameters in **Figure 8.14**. The scalar gain k^* is defined to be

$$k^* = \frac{k_m}{k_p}$$

and is intended to modulate the high-frequency gain of the control system. The vector $\theta_1{}^*$ contains $(n-1)$ parameters which intend to cancel the zeros of the plant. The vector $\theta_2{}^*$ contains $(n-1)$ parameters which, together with the scalar gain $\theta_o{}^*$ can move the poles of the closed-loop control system to the locations of the reference model poles. Comparing Figure 8.13 and Figure 8.14 will help the reader become familiar with this structure and the corresponding notations.

As before, the control input in this system is a linear combination of the reference signal $r(t)$, the vector signal ω_1 obtained by filtering the control input u, the signals ω_2 obtained by filtering the plant output y, and the output itself. The control input u can thus be written, in terms of the adjustable parameters and the various signals, as

$$u^*(t) = k^* r + \theta_1{}^* \omega_1 + \theta_2{}^* \omega_2 + \theta_o{}^* y \qquad (8.49)$$

Corresponding to this control law and *any* reference input $r(t)$, the output of the plant is

$$y(t) = \frac{B(p)}{A(p)} u^*(t) = W_m r(t) \qquad (8.50)$$

since these parameters result in perfect tracking. At this point, one easily sees the reason for assuming the plant to be minimum-phase: this allows the plant zeros to be canceled by the controller poles.

In the adaptive control problem, the plant parameters are unknown, and the ideal control parameters described above are also unknown. Instead of (8.49), the control law is chosen to be

$$u = k(t) r + \theta_1(t) \omega_1 + \theta_2(t) \omega_2 + \theta_o(t) y \qquad (8.51)$$

where $k(t)$, $\theta_1(t)$, $\theta_2(t)$ and $\theta_o(t)$ are controller parameters to be provided by the adaptation law.

Example 8.6: Consider the second order plant described by the transfer function

$$y = \frac{k_p u}{p^2 + a_{p1} p + a_{p2}}$$

and the reference model

$$y_m = \frac{k_m r}{p^2 + a_{m1} p + a_{m2}}$$

which are similar to those in Example 8.5, but now contain no zeros.

Let us consider the control structure shown in Figure 8.16 which is a slight modification of the controller structure in Figure 8.13. Note that b_{m1} in the filters in Figure 8.13 has been replaced by a positive number λ_o. Of course, the transfer functions W_p and W_m in Figure 8.16 now have relative degree 2.

The closed-loop transfer function from the reference signal r to the plant output y is

$$W_{ry} = k \frac{\dfrac{p+\lambda_o}{p+\lambda_o+\alpha_1} \dfrac{k_p}{p^2 + a_{p1}p + a_{p2}}}{1 + \dfrac{p+\lambda_o}{p+\lambda_o+\alpha_1} \dfrac{\beta_1 p + \beta_2}{p+\lambda_o} \dfrac{k_p}{p^2 + a_{p1}p + a_{p2}}}$$

$$= \frac{k k_p (p+\lambda_o)}{(p+\lambda_o+\alpha_1)(p^2 + a_{p1}p + a_{p2}) + k_p(\beta_1 p + \beta_2)}$$

Therefore, if the controller parameters α_1, β_1, β_2, and k are chosen such that

$$(p+\lambda_o+\alpha_1)(p^2 + a_{p1}p + a_{p2}) + k_p(\beta_1 p + \beta_2) = (p+\lambda_o)(p^2 + a_{m1}p + a_{m2})$$

and

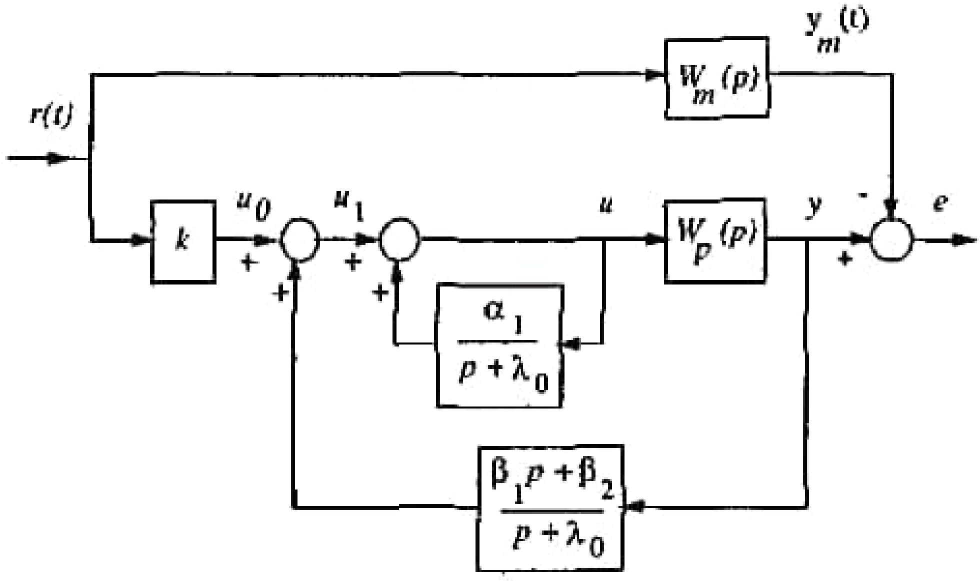

Figure 8.16 : A model-reference control system for relative degree 2

$$k = \frac{k_m}{k_p}$$

then the closed loop transfer function W_{ry} becomes identically the same as that of the reference model. Clearly, such choice of parameters exists and is *unique*. □

For general plants of relative degree larger than 1, the same control structure as given in Figure 8.14 is chosen. Note that the order of the filters in the control law is still $(n-1)$. However, since the model numerator polynomial $Z_m(p)$ is of degree smaller than $(n-1)$, it is no longer possible to choose the poles of the filters in the controller so that $\det[p\mathbf{I} - \Lambda] = Z_m(p)$ as in (8.48). Instead, we now choose

$$\lambda(p) = Z_m(p)\, \lambda_1(p) \tag{8.57}$$

where $\lambda(p) = \det[p\mathbf{I} - \Lambda]$ and $\lambda_1(p)$ is a Hurwitz polynomial of degree $(n-1-m)$. With this choice, the desired zeros of the reference model can be imposed.

Let us denote the transfer function of the feedforward part (u/u_1) of the controller by $\lambda(p)/(\lambda(p) + C(p))$, and that of the feedback part by $D(p)/\lambda(p)$, where the polynomial $C(p)$ contains the parameters in the vector θ_1, and the polynomial $D(p)$ contains θ_o and the parameters in the vector θ_2. Then, the closed-loop transfer function is easily found to be

$$W_{ry} = \frac{k k_p Z_p \lambda_1(p) Z_m(p)}{R_p(p)[\lambda(p) + C(p)] + k_p Z_p D(p)} \tag{8.58}$$

The question now is whether in this general case there exist choice of values for k, θ_o, θ_1 and θ_2 such that the above transfer function becomes exactly the same as $W_m(p)$, or equivalently

$$R_p(\lambda(p) + C(p)) + k_p Z_p D(p) = \lambda_1 Z_p R_m(p) \tag{8.59}$$

The answer to this question can be obtained from the following lemma:

Lemma 8.2: *Let $A(p)$ and $B(p)$ be polynomials of degree n_1 and n_2, respectively. If $A(p)$ and $B(p)$ are relatively prime, then there exist polynomials $M(p)$ and $N(p)$ such that*

$$A(p)M(p) + B(p)N(p) = A^*(p) \tag{8.60}$$

where $A^*(p)$ is an arbitrary polynomial.

This lemma can be used straightforwardly to answer our question regarding (8.59). By regarding R_p as $A(p)$ in the lemma, $k_p Z_p$ as $B(P)$ and $\lambda_1(p) Z_p R_m$ as $A^*(p)$, we conclude that there exist polynomials $(\lambda(p) + C(p))$ and $D(p)$ such that (8.59) is satisfied. This implies that a proper choice of the controller parameters

$$k = k^* \qquad \theta_o = \theta_o^* \qquad \theta_1 = \theta_1^* \qquad \theta_2 = \theta_2^*$$

exists so that exact model-following is achieved.

Example 8.7: Rohrs's Example

The sometimes destructive consequence of non-parametric uncertainties is clearly shown in the well-known example by Rohrs, which consists of an adaptive first-order control system containing unmodeled dynamics and measurement noise. In the adaptive control design, the plant is assumed to have a the following nominal model

$$H_o(p) = \frac{k_p}{p + a_p}$$

The reference model has the following SPR function

$$M(p) = \frac{k_m}{p + a_m} = \frac{3}{p + 3}$$

The real plant, however, is assumed to have the transfer function relation

$$y = \frac{2}{p+1}\,\frac{229}{p^2 + 30p + 229}\,u$$

This means that the real plant is of third order while the nominal plant is of only first order. The unmodeled dynamics are thus seen to be $229/(p^2 + 30p + 229)$, which are high-frequency but *lightly-damped* poles at $(-15 + j)$ and $(-15 - j)$.

Besides the unmodeled dynamics, it is assumed that there is some measurement noise $n(t)$ in the adaptive system. The whole adaptive control system is shown in Figure 8.18. The measurement noise is assumed to be $n(t) = 0.5\sin(16.1\,t)$.

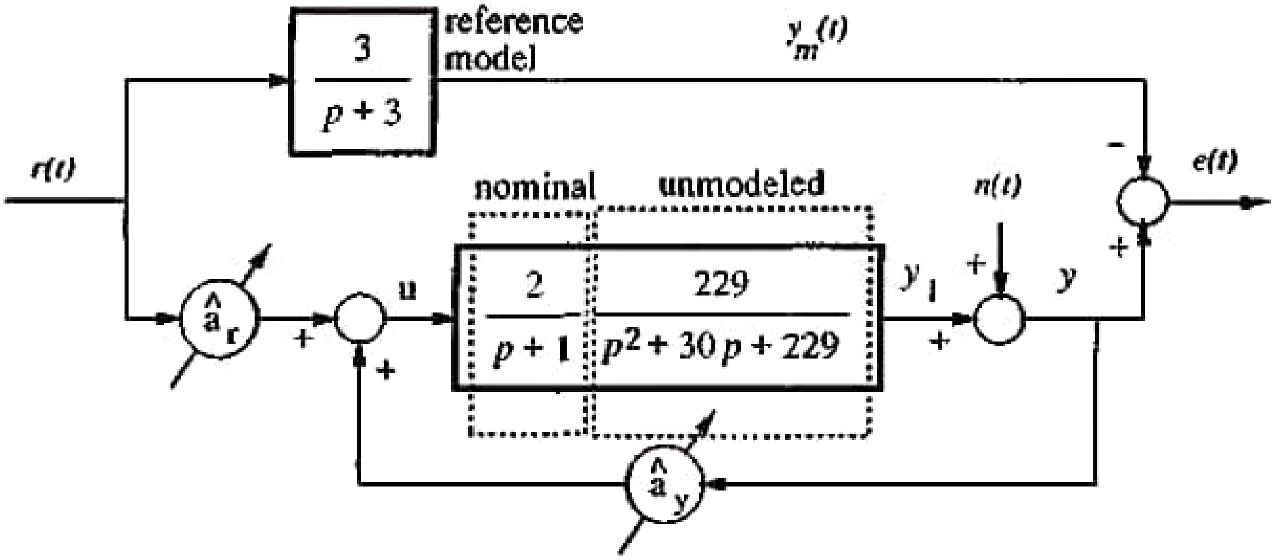

Figure 8.18 : Adaptive control with unmodeled dynamics and measurement noise

Corresponding to the reference input $r(t) = 2$, the results of the adaptive control system are shown in Figure 8.19. It is seen that the output $y(t)$ initially converges to the vicinity of $y = 2$, then operates with a small oscillatory error related to the measurement noise, and finally diverges to infinity. □

In view of the *global* tracking convergence proven in the absence of non-parametric uncertainties and the *small* amount of non-parametric uncertainties present in the above example, the observed instability can seem quite surprising. However, one can gain some insight into what is going on in the adaptive control system by examining the parameter estimates in Figure 8.19. It is seen that the parameters drift slowly as time goes on, and suddenly diverge sharply. The simplest explanation of the parameter drift problem is that the constant reference input contains insufficient parameter information and the parameter adaptation mechanism has difficulty distinguishing the parameter information from noise. As a result, the parameters drift in a direction along which the tracking error remains small. Note that even though the

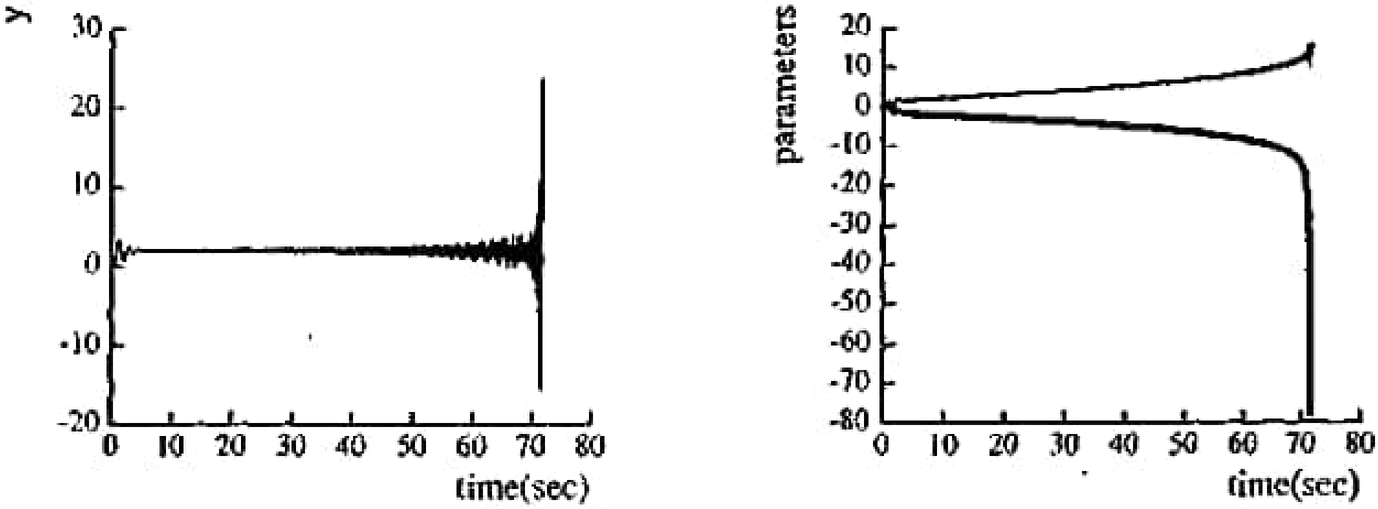

Figure 8.19 : Instability and parameter drift

tracking error stays at the same level when the parameters drift, the poles of the closed-loop system continuously shift (since the parameters vary very slowly, the adaptive control system may be regarded as a linear time-invariant system with three poles). When the estimated parameters drift to the point where the closed-loop poles enter the right-half complex plane, the whole system becomes unstable. The above reasoning can be confirmed mathematically.

In general, the following points can be made about parameter drift. Parameter drift occurs when the signals are not persistently exciting; it is mainly caused by measurement noise; it does not affect tracking accuracy until instability occurs; it leads to sudden failure of the adaptive control system (by exciting unmodeled dynamics).

Parameter drift is a major problem associated with non-parametric uncertainties (noise and disturbance). But there are possibly other problems. For example, when the adaptation gain or the reference signal are very large, adaptation becomes fast and the estimated parameters may be quite oscillatory. If the oscillations get into the frequency range of unmodeled dynamics, the unmodeled dynamics may be excited and the parameter adaptation may be based on meaningless signals, possibly leading to instability of the control system. For parameter oscillation problems, techniques such as normalization of signals (divide $\mathbf{v}$ by $1 + \mathbf{v}^T \mathbf{v}$) or the composite adaptation in section 8.8 can be quite useful.