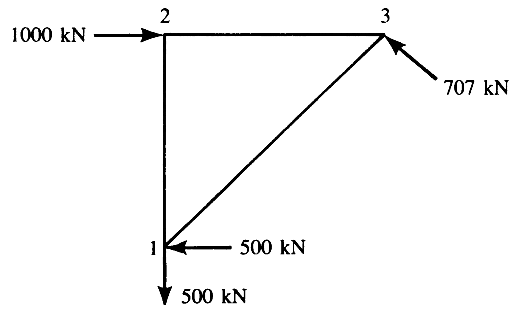


Figure 3–24 Free-body diagram of the truss of Figure 3–23

Boundary elements are used to specify nonzero displacements and rotations to nodes. They are also used to evaluate reactions at rigid and flexible supports. Boundary elements are two-node elements. The line defined by the two nodes specifies the direction along which the force reaction is evaluated or the displacement is specified. In the case of moment reaction, the line specifies the axis about which the moment is evaluated and the rotation is specified.

We consider boundary elements that are used to obtain reaction forces (rigid boundary elements) or specify translational displacements (displacement boundary elements) as truss elements with only one nonzero translational stiffness. Boundary elements used to either evaluate reaction moments or specify rotations behave like beam elements with only one nonzero stiffness corresponding to the rotational stiffness about the specified axis.

The elastic boundary elements are used to model flexible supports and to calculate reactions at skewed or inclined boundaries. Consult Reference [9] for more details about using boundary elements.

3.10 Potential Energy Approach to Derive Bar Element Equations

We now present the principle of minimum potential energy to derive the bar element equations. Recall from Section 2.6 that the total potential energy π_p was defined as the sum of the internal strain energy U and the potential energy of the external forces Ω:

$$\pi_p = U + \Omega \tag{3.10.1}$$

To evaluate the strain energy for a bar, we consider only the work done by the internal forces during deformation. Because we are dealing with a one-dimensional bar, the internal force doing work is given in Figure 3–25 as $\sigma_x(\Delta y)(\Delta z)$, due only to normal stress σ_x. The displacement of the x face of the element is $\Delta x(\varepsilon_x)$; the displacement of the $x + \Delta x$ face is $\Delta x(\varepsilon_x + d\varepsilon_x)$. The change in displacement is then

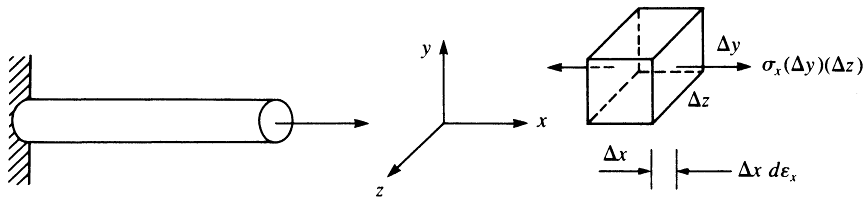


Figure 3–25 Internal force in a one-dimensional bar

$\Delta x\, d\varepsilon_x$, where $d\varepsilon_x$ is the differential change in strain occurring over length Δx. The differential internal work (or strain energy) dU is the internal force multiplied by the displacement through which the force moves, given by

$$dU = \sigma_x(\Delta y)(\Delta z)(\Delta x)\, d\varepsilon_x \tag{3.10.2}$$

Rearranging and letting the volume of the element approach zero, we obtain, from Eq. (3.10.2),

$$dU = \sigma_x\, d\varepsilon_x\, dV \tag{3.10.3}$$

For the whole bar, we then have

$$U = \iiint_V \left\{ \int_0^{\varepsilon_x} \sigma_x\, d\varepsilon_x \right\} dV \tag{3.10.4}$$

Now, for a linear-elastic (Hooke's law) material as shown in Figure 3–26, we see that $\sigma_x = E\varepsilon_x$. Hence substituting this relationship into Eq. (3.10.4), integrating with respect to ε_x, and then resubstituting σ_x for $E\varepsilon_x$, we have

$$U = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x\, dV \tag{3.10.5}$$

as the expression for the strain energy for one-dimensional stress.

The potential energy of the external forces, being opposite in sign from the external work expression because the potential energy of external forces is lost when the

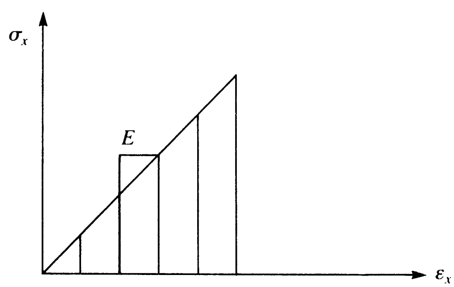


Figure 3–26 Linear-elastic (Hooke's law) material

work is done by the external forces, is given by

$$\Omega = -\iiint_V \hat{X}_b \hat{u}\, dV - \iint_{S_1} \hat{T}_x \hat{u}_s\, dS - \sum_{i=1}^{M} \hat{f}_{ix} \hat{d}_{ix} \tag{3.10.6}$$

where the first, second, and third terms on the right side of Eq. (3.10.6) represent the potential energy of (1) body forces $\hat{X}_b$, typically from the self-weight of the bar (in units of force per unit volume) moving through displacement function $\hat{u}$, (2) surface loading or traction $\hat{T}_x$, typically from distributed loading acting along the surface of the element (in units of force per unit surface area) moving through displacements $\hat{u}_s$, where $\hat{u}_s$ are the displacements occurring over surface S_1, and (3) nodal concentrated forces $\hat{f}_{ix}$ moving through nodal displacements $\hat{d}_{ix}$. The forces $\hat{X}_b$, $\hat{T}_x$, and $\hat{f}_{ix}$ are considered to act in the local $\hat{x}$ direction of the bar as shown in Figure 3–27. In Eqs. (3.10.5) and (3.10.6), V is the volume of the body and S_1 is the part of the surface S on which surface loading acts. For a bar element with two nodes and one degree of freedom per node, $M = 2$.

We are now ready to describe the finite element formulation of the bar element equations by using the principle of minimum potential energy.

The finite element process seeks a minimum in the potential energy within the constraint of an assumed displacement pattern within each element. The greater the number of degrees of freedom associated with the element (usually meaning increasing the number of nodes), the more closely will the solution approximate the true one and ensure complete equilibrium (provided the true displacement can, in the limit, be approximated). An approximate finite element solution found by using the stiffness method will always provide an approximate value of potential energy greater than or equal to the correct one. This method also results in a structure behavior that is predicted to be physically stiffer than, or at best to have the same stiffness as, the actual one. This is explained by the fact that the structure model is allowed to displace only into shapes defined by the terms of the assumed displacement field within each element of the structure. The correct shape is usually only approximated by the assumed field, although the correct shape can be the same as the assumed field. The assumed field effectively constrains the structure from deforming in its natural manner. This constraint effect stiffens the predicted behavior of the structure.

Apply the following steps when using the principle of minimum potential energy to derive the finite element equations.

1. Formulate an expression for the total potential energy.
2. Assume the displacement pattern to vary with a finite set of undetermined parameters (here these are the nodal displacements d_{ix}), which are substituted into the expression for total potential energy.
3. Obtain a set of simultaneous equations minimizing the total potential energy with respect to these nodal parameters. These resulting equations represent the element equations.

The resulting equations are the approximate (or possibly exact) equilibrium equations whose solution for the nodal parameters seeks to minimize the potential energy when back-substituted into the potential energy expression. The preceding

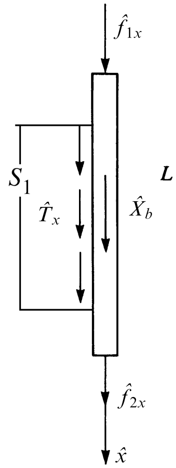


Figure 3–27 General forces acting on a one-dimensional bar

three steps will now be followed to derive the bar element equations and stiffness matrix.

Consider the bar element of length L, with constant cross-sectional area A, shown in Figure 3–27. Using Eqs. (3.10.5) and (3.10.6), we find that the total potential energy, Eq. (3.10.1), becomes

$$\pi_p = \frac{A}{2}\int_0^L \sigma_x \varepsilon_x \, d\hat{x} - \hat{f}_{1x}\hat{d}_{1x} - \hat{f}_{2x}\hat{d}_{2x} - \iint_{S_1} \hat{u}_s \hat{T}_x \, dS - \iiint_V \hat{u}\hat{X}_b \, dV \tag{3.10.7}$$

because A is a constant and variables σ_x and ε_x at most vary with $\hat{x}$.

From Eqs. (3.1.3) and (3.1.4), we have the axial displacement function expressed in terms of the shape functions and nodal displacements by

$$\hat{u} = [N]\{\hat{d}\} \qquad \hat{u}_s = [N_S]\{\hat{d}\} \tag{3.10.8}$$

where

$$[N] = \left[1 - \frac{\hat{x}}{L} \quad \frac{\hat{x}}{L}\right] \tag{3.10.9}$$

$[N_S]$ is the shape function matrix evaluated over the surface that the distributed surface traction acts and

$$\{\hat{d}\} = \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} \tag{3.10.10}$$

Then, using the strain/displacement relationship $\varepsilon_x = d\hat{u}/d\hat{x}$, we can write the axial strain as

$$\{\varepsilon_x\} = \left[-\frac{1}{L} \quad \frac{1}{L}\right]\{\hat{d}\} \tag{3.10.11}$$

or

$$\{\varepsilon_x\} = [B]\{\hat{d}\} \tag{3.10.12}$$

where we define

$$[B] = \left[-\frac{1}{L} \quad \frac{1}{L}\right] \tag{3.10.13}$$

The axial stress/strain relationship is given by

$$\{\sigma_x\} = [D]\{\varepsilon_x\} \tag{3.10.14}$$

where

$$[D] = [E] \tag{3.10.15}$$

for the one-dimensional stress/strain relationship and E is the modulus of elasticity. Now, by Eq. (3.10.12), we can express Eq. (3.10.14) as

$$\{\sigma_x\} = [D][B]\{\hat{d}\} \tag{3.10.16}$$

Using Eq. (3.10.7) expressed in matrix notation form, we have the total potential energy given by

$$\pi_p = \frac{A}{2}\int_0^L \{\sigma_x\}^T\{\varepsilon_x\}\,d\hat{x} - \{\hat{d}\}^T\{P\} - \iint_{S_1} \{\hat{u}_s\}^T\{\hat{T}_x\}\,dS - \iiint_V \{\hat{u}\}^T\{\hat{X}_b\}\,dV \tag{3.10.17}$$

where $\{P\}$ now represents the concentrated nodal loads and where in general both $\underline{\sigma}_x$ and $\underline{\varepsilon}_x$ are column matrices. For proper matrix multiplication, we must place the transpose on $\{\sigma_x\}$. Similarly, $\{\hat{u}\}$ and $\{\hat{T}_x\}$ in general are column matrices, so for proper matrix multiplication, $\{\hat{u}\}$ is transposed in Eq. (3.10.17).

Using Eqs. (3.10.8), (3.10.12), and (3.10.16) in Eq. (3.10.17), we obtain

$$\begin{aligned}\pi_p = {} & \frac{A}{2}\int_0^L \{\hat{d}\}^T[B]^T[D]^T[B]\{\hat{d}\}\,d\hat{x} - \{\hat{d}\}^T\{P\} \\ & - \iint_{S_1} \{\hat{d}\}^T[N_S]^T\{\hat{T}_x\}\,dS - \iiint_V \{\hat{d}\}^T[N]^T\{\hat{X}_b\}\,dV\end{aligned} \tag{3.10.18}$$

In Eq. (3.10.18), π_p is seen to be a function of $\{\hat{d}\}$; that is, $\pi_p = \pi_p(\hat{d}_{1x}, \hat{d}_{2x})$. However, $[B]$ and $[D]$, Eqs. (3.10.13) and (3.10.15), and the nodal degrees of freedom $\hat{d}_{1x}$ and $\hat{d}_{2x}$ are not functions of $\hat{x}$. Therefore, integrating Eq. (3.10.18) with respect to $\hat{x}$ yields

$$\pi_p = \frac{AL}{2}\{\hat{d}\}^T[B]^T[D]^T[B]\{\hat{d}\} - \{\hat{d}\}^T\{\hat{f}\} \tag{3.10.19}$$

where

$$\{\hat{f}\} = \{P\} + \iint_{S_1} [N_S]^T\{\hat{T}_x\}\,dS + \iiint_V [N]^T\{\hat{X}_b\}\,dV \tag{3.10.20}$$

From Eq. (3.10.20), we observe three separate types of load contributions from concentrated nodal forces, surface tractions, and body forces, respectively. We define these surface tractions and body-force matrices as

$$\{\hat{f}_s\} = \iint_{S_1} [N_S]^T \{\hat{T}_x\}\, dS \tag{3.10.20a}$$

$$\{\hat{f}_b\} = \iiint_{V} [N]^T \{\hat{X}_b\}\, dV \tag{3.10.20b}$$

The expression for $\{\hat{f}\}$ given by Eq. (3.10.20) then describes how certain loads can be considered to best advantage.

Loads calculated by Eqs. (3.10.20a) and (3.10.20b) are called consistent because they are based on the same shape functions $[N]$ used to calculate the element stiffness matrix. The loads calculated by Eq. (3.10.20a) and (3.10.20b) are also statically equivalent to the original loading; that is, both $\{\hat{f}_s\}$ and $\{\hat{f}_b\}$ and the original loads yield the same resultant force and same moment about an arbitrarily chosen point.

The minimization of π_p with respect to each nodal displacement requires that

$$\frac{\partial \pi_p}{\partial \hat{d}_{1x}} = 0 \qquad \text{and} \qquad \frac{\partial \pi_p}{\partial \hat{d}_{2x}} = 0 \tag{3.10.21}$$

Now we explicitly evaluate π_p given by Eq. (3.10.19) to apply Eq. (3.10.21). We define the following for convenience:

$$\{U^*\} = \{\hat{d}\}^T [B]^T [D]^T [B] \{\hat{d}\} \tag{3.10.22}$$

Using Eqs. (3.10.10), (3.10.13), and (3.10.15) in Eq. (3.10.22) yields

$$\{U^*\} = [\hat{d}_{1x} \quad \hat{d}_{2x}] \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} [E] \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} \tag{3.10.23}$$

Simplifying Eq. (3.10.23), we obtain

$$U^* = \frac{E}{L^2} (\hat{d}_{1x}^2 - 2\hat{d}_{1x}\hat{d}_{2x} + \hat{d}_{2x}^2) \tag{3.10.24}$$

Also, the explicit expression for $\{\hat{d}\}^T \{\hat{f}\}$ is

$$\{\hat{d}\}^T \{\hat{f}\} = \hat{d}_{1x}\hat{f}_{1x} + \hat{d}_{2x}\hat{f}_{2x} \tag{3.10.25}$$

Therefore, using Eqs. (3.10.24) and (3.10.25) in Eq. (3.10.19) and then applying Eqs. (3.10.21), we obtain

$$\frac{\partial \pi_p}{\partial \hat{d}_{1x}} = \frac{AL}{2} \left[\frac{E}{L^2} (2\hat{d}_{1x} - 2\hat{d}_{2x}) \right] - \hat{f}_{1x} = 0 \tag{3.10.26}$$

and $$\frac{\partial \pi_p}{\partial \hat{d}_{2x}} = \frac{AL}{2}\left[\frac{E}{L^2}(-2\hat{d}_{1x} + 2\hat{d}_{2x})\right] - \hat{f}_{2x} = 0$$

In matrix form, we express Eqs. (3.10.26) as

$$\frac{\partial \pi_p}{\partial \{\hat{d}\}} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} - \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.10.27)$$

or, because $\{\hat{f}\} = [\hat{k}]\{\hat{d}\}$, we have the stiffness matrix for the bar element obtained from Eq. (3.10.27) as

$$[\hat{k}] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (3.10.28)$$

As expected, Eq. (3.10.28) is identical to the stiffness matrix obtained in Section 3.1.

Finally, instead of the cumbersome process of explicitly evaluating π_p, we can use the matrix differentiation as given by Eq. (2.6.12) and apply it directly to Eq. (3.10.19) to obtain

$$\frac{\partial \pi_p}{\partial \{\hat{d}\}} = AL[B]^T[D][B]\{\hat{d}\} - \{\hat{f}\} = 0 \quad (3.10.29)$$

where $[D]^T = [D]$ has been used in writing Eq. (3.10.29). The result of the evaluation of $AL[B]^T[D][B]$ is then equal to $[\hat{k}]$ given by Eq. (3.10.28). Throughout this text, we will use this matrix differentiation concept (also see Appendix A), which greatly simplifies the task of evaluating $[\hat{k}]$.

To illustrate the use of Eq. (3.10.20a) to evaluate the equivalent nodal loads for a bar subjected to axial loading traction $\hat{T}_x$, we now solve Example 3.12.

Example 3.12

A bar of length L is subjected to a linearly distributed axial loading that varies from zero at node 1 to a maximum at node 2 (Figure 3–28). Determine the energy equivalent nodal loads.

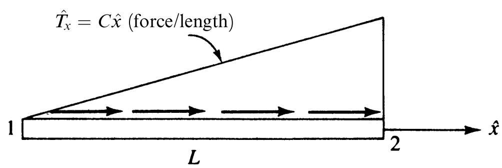


Figure 3–28 Element subjected to linearly varying axial load

Using Eq. (3.10.20a) and shape functions from Eq. (3.10.9), we solve for the energy equivalent nodal forces of the distributed loading as follows:

$$\{\hat{f}_0\} = \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \iint_{S_1} [N]^T \{\hat{T}_x\}\, dS = \int_0^L \begin{Bmatrix} 1 - \dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} \{C\hat{x}\}\, d\hat{x} \tag{3.10.30}$$

$$= \begin{Bmatrix} \dfrac{C\hat{x}^2}{2} - \dfrac{C\hat{x}^3}{3L} \\ \dfrac{C\hat{x}^3}{3L} \end{Bmatrix}_0^L$$

$$= \begin{Bmatrix} \dfrac{CL^2}{6} \\ \dfrac{CL^2}{3} \end{Bmatrix} \tag{3.10.31}$$

where the integration was carried out over the length of the bar, because $\hat{T}_x$ is in units of force/length.

Note that the total load is the area under the load distribution given by

$$F = \frac{1}{2}(L)(CL) = \frac{CL^2}{2} \tag{3.10.32}$$

Therefore, comparing Eq. (3.10.31) with (3.10.32), we find that the equivalent nodal loads for a linearly varying load are

$$\begin{aligned} \hat{f}_{1x} &= \frac{1}{3}F = \text{one-third of the total load} \\ \hat{f}_{2x} &= \frac{2}{3}F = \text{two-thirds of the total load} \end{aligned} \tag{3.10.33}$$

In summary, for the simple two-noded bar element subjected to a linearly varying load (triangular loading), place one-third of the total load at the node where the distributed loading begins (zero end of the load) and two-thirds of the total load at the node where the peak value of the distributed load ends. ■

We now illustrate (Example 3.13) a complete solution for a bar subjected to a surface traction loading.

Example 3.13

For the rod loaded axially as shown in Figure 3–29, determine the axial displacement and axial stress. Let $E = 30 \times 10^6$ psi, $A = 2$ in.^{2}, and $L = 60$ in. Use (a) one and (b) two elements in the finite element solutions. (In Section 3.11 one-, two-, four-, and eight-element solutions will be presented from the computer program Algor [9].

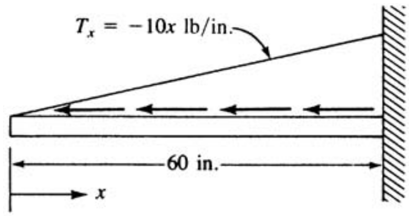


Figure 3–29 Rod subjected to triangular load distribution

(a) One-element solution (Figure 3–30).

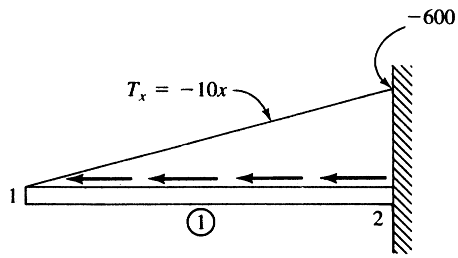


Figure 3–30 One-element model

From Eq. (3.10.20a), the distributed load matrix is evaluated as follows:

$$\{F_0\} = \int_0^L [N]^T \{T_x\}\, dx \tag{3.10.34}$$

where T_x is a line load in units of pounds per inch and $\underline{\hat{f}}_0 = \underline{F}_0$ as $\underline{x} = \underline{\hat{x}}$. Therefore, using Eq. (3.1.4) for $[N]$ in Eq. (3.10.34), we obtain

$$\{F_0\} = \int_0^L \begin{Bmatrix} 1 - \dfrac{x}{L} \\ \dfrac{x}{L} \end{Bmatrix} \{-10x\}\, dx \tag{3.10.35}$$

or

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \end{Bmatrix} = \begin{Bmatrix} \dfrac{-10L^2}{2} + \dfrac{10L^2}{3} \\ \dfrac{-10L^2}{3} \end{Bmatrix} = \begin{Bmatrix} \dfrac{-10L^2}{6} \\ \dfrac{-10L^2}{3} \end{Bmatrix} = \begin{Bmatrix} \dfrac{-10(60)^2}{6} \\ \dfrac{-10(60)^2}{3} \end{Bmatrix}$$

or

$$F_{1x} = -6000 \text{ lb} \qquad F_{2x} = -12{,}000 \text{ lb} \tag{3.10.36}$$

Using Eq. (3.10.33), we could have determined the same forces at nodes 1 and 2—that is, one-third of the total load is at node 1 and two-thirds of the total load is at node 2.

Using Eq. (3.10.28), we find that the stiffness matrix is given by

$$k^{(1)} = 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The element equations are then

$$10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -6000 \\ R_{2x} - 12{,}000 \end{Bmatrix} \tag{3.10.37}$$

Solving Eq. 1 of Eq. (3.10.37), we obtain

$$d_{1x} = -0.006 \text{ in.} \tag{3.10.38}$$

The stress is obtained from Eq. (3.10.14) as

$$\begin{aligned} \{\sigma_x\} &= [D]\{\varepsilon_x\} \\ &= E[B]\{d\} \\ &= E\begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix} \\ &= E\left(\frac{d_{2x} - d_{1x}}{L}\right) \\ &= 30 \times 10^6 \left(\frac{0 + 0.006}{60}\right) \\ &= 3000 \text{ psi } (T) \end{aligned} \tag{3.10.39}$$

(b) Two-element solution (Figure 3–31).

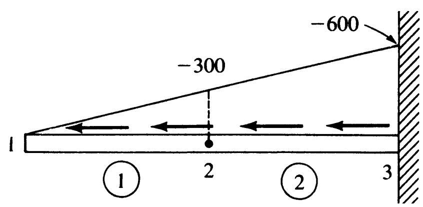


Figure 3–31 Two-element model

We first obtain the element forces. For element 2, we divide the load into a uniform part and a triangular part. For the uniform part, half the total uniform load is placed at each node associated with the element. Therefore, the total uniform part is

$$(30 \text{ in.})(-300 \text{ lb/in.}) = -9000 \text{ lb}$$

and using Eq. (3.10.33) for the triangular part of the load, we have, for element 2,

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -[\frac{1}{2}(9000) + \frac{1}{3}(4500)] \\ -[\frac{1}{2}(9000) + \frac{2}{3}(4500)] \end{Bmatrix} = \begin{Bmatrix} -6000 \text{ lb} \\ -7500 \text{ lb} \end{Bmatrix} \tag{3.10.40}$$

For element 1, the total force is from the triangle-shaped distributed load only and is given by

$$\tfrac{1}{2}(30 \text{ in.})(-300 \text{ lb/in.}) = -4500 \text{ lb}$$

On the basis of Eq. (3.10.33), this load is separated into nodal forces as shown:

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \tfrac{1}{3}(-4500) \\ \tfrac{2}{3}(-4500) \end{Bmatrix} = \begin{Bmatrix} -1500 \text{ lb} \\ -3000 \text{ lb} \end{Bmatrix} \tag{3.10.41}$$

The final nodal force matrix is then

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{Bmatrix} -1500 \\ -6000 - 3000 \\ R_{3x} - 7500 \end{Bmatrix} \tag{3.10.42}$$

The element stiffness matrices are now

$$\underline{k}^{(1)} = \underline{k}^{(2)} = \frac{AE}{L/2} \overset{\begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix}}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} = (2 \times 10^6) \overset{\begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix}}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \tag{3.10.43}$$

The assembled global stiffness matrix is

$$\underline{K} = (2 \times 10^6) \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \frac{\text{lb}}{\text{in.}} \tag{3.10.44}$$

The assembled global equations are then

$$(2 \times 10^6) \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} = 0 \end{Bmatrix} = \begin{Bmatrix} -1500 \\ -9000 \\ R_{3x} - 7500 \end{Bmatrix} \tag{3.10.45}$$

where the boundary condition $d_{3x} = 0$ has been substituted into Eq. (3.10.45). Now, solving equations 1 and 2 of Eq. (3.10.45), we obtain

$$\begin{aligned} d_{1x} &= -0.006 \text{ in.} \\ d_{2x} &= -0.00525 \text{ in.} \end{aligned} \tag{3.10.46}$$

The element stresses are as follows:

Element 1

$$\begin{aligned} \sigma_x &= E\begin{bmatrix} -\frac{1}{30} & \frac{1}{30} \end{bmatrix} \begin{Bmatrix} d_{1x} = -0.006 \\ d_{2x} = -0.00525 \end{Bmatrix} \\ &= 750 \text{ psi } (T) \end{aligned} \tag{3.10.47}$$

Element 2

$$\sigma_x = E\left[-\frac{1}{30} \quad \frac{1}{30}\right]\begin{Bmatrix} d_{2x} = -0.00525 \\ d_{3x} = 0 \end{Bmatrix}$$

$$= 5250 \text{ psi } (T) \tag{3.10.48}$$

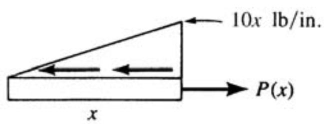
■

3.11 Comparison of Finite Element Solution to Exact Solution for Bar

We will now compare the finite element solutions for Example 3.13 using one, two, four, and eight elements to model the bar element and the exact solution. The exact solution for displacement is obtained by solving the equation

$$u = \frac{1}{AE}\int_0^x P(x)\,dx \tag{3.11.1}$$

where, using the following free-body diagram,



we have

$$P(x) = \tfrac{1}{2}x(10x) = 5x^2 \text{ lb} \tag{3.11.2}$$

Therefore, substituting Eq. (3.11.2) into Eq. (3.11.1), we have

$$u = \frac{1}{AE}\int_0^x 5x^2\,dx$$

$$= \frac{5x^3}{3AE} + C_1 \tag{3.11.3}$$

Now, applying the boundary condition at $x = L$, we obtain

$$u(L) = 0 = \frac{5L^3}{3AE} + C_1$$

or

$$C_1 = -\frac{5L^3}{3AE} \tag{3.11.4}$$

Substituting Eq. (3.11.4) into Eq. (3.11.3) makes the final expression for displacement

$$u = \frac{5}{3AE}(x^3 - L^3) \tag{3.11.5}$$

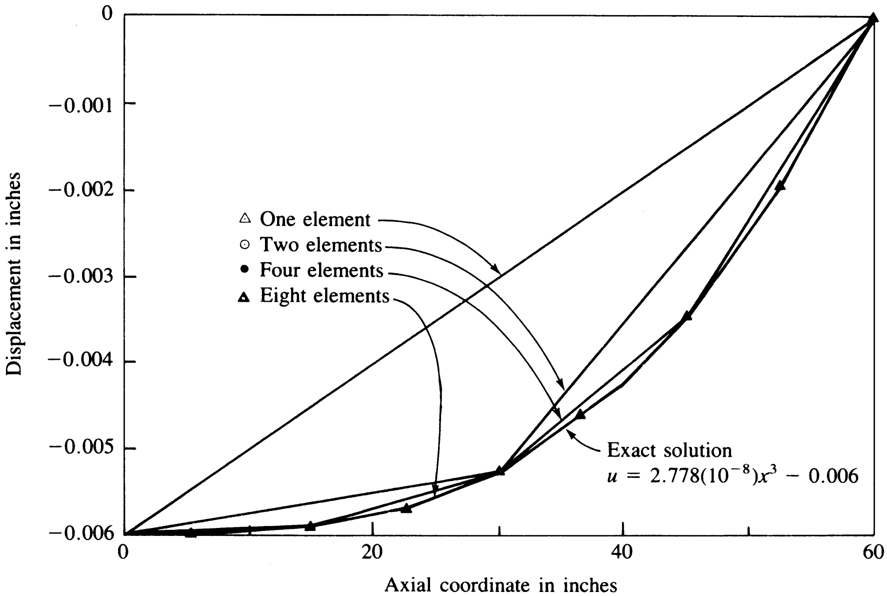


Figure 3–32 Comparison of exact and finite element solutions for axial displacement (along length of bar)

Substituting $A = 2$ in.^{2}, $E = 30 \times 10^6$ psi, and $L = 60$ in. into Eq. (3.11.5), we obtain

$$u = 2.778 \times 10^{-8} x^3 - 0.006 \tag{3.11.6}$$

The exact solution for axial stress is obtained by solving the equation

$$\sigma(x) = \frac{P(x)}{A} = \frac{5x^2}{2 \text{ in}^2} = 2.5x^2 \text{ psi} \tag{3.11.7}$$

Figure 3–32 shows a plot of Eq. (3.11.6) along with the finite element solutions (part of which were obtained in Example 3.13). Some conclusions from these results follow.

1. The finite element solutions match the exact solution at the node points. The reason why these nodal values are correct is that the element nodal forces were calculated on the basis of being energy-equivalent to the distributed load based on the assumed linear displacement field within each element. (For uniform cross-sectional bars and beams, the nodal degrees of freedom are exact. In general, computed nodal degrees of freedom are not exact.)
2. Although the node values for displacement match the exact solution, the values at locations between the nodes are poor using few elements (see one- and two-element solutions) because we used a linear displacement function within each element, whereas the exact solution, Eq. (3.11.6), is a cubic function. However, because we use increasing

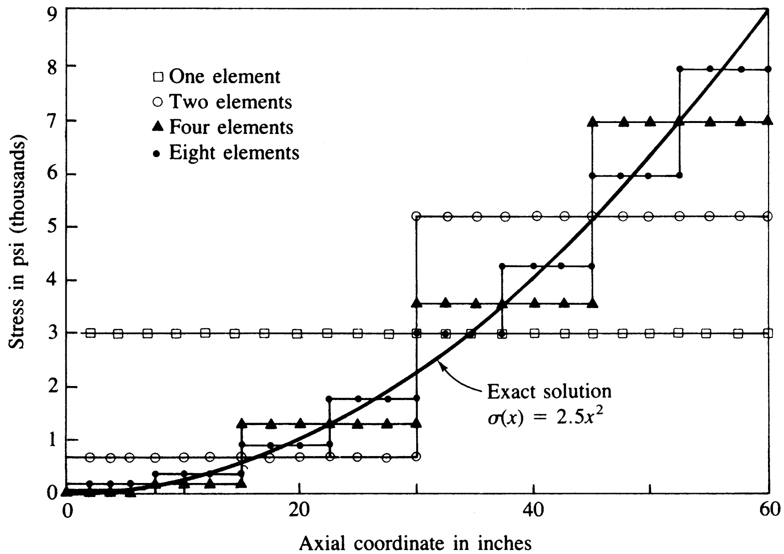


Figure 3–33 Comparison of exact and finite element solutions for axial stress (along length of bar)

numbers of elements, the finite element solution converges to the exact solution (see the four- and eight-element solutions in Figure 3–32).

3. The stress is derived from the slope of the displacement curve as $\sigma = E\varepsilon = E(du/dx)$. Therefore, by the finite element solution, because u is a linear function in each element, axial stress is constant in each element. It then takes even more elements to model the first derivative of the displacement function or, equivalently, the axial stress. This is shown in Figure 3–33, where the best results occur for the eight-element solution.
4. The best approximation of the stress occurs at the midpoint of the element, not at the nodes (Figure 3–33). This is because the derivative of displacement is better predicted between the nodes than at the nodes.
5. The stress is not continuous across element boundaries. Therefore, equilibrium is not satisfied across element boundaries. Also, equilibrium within each element is, in general, not satisfied. This is shown in Figure 3–34 for element 1 in the two-element solution and element 1 in the eight-element solution [in the eight-element solution the forces are obtained from the Algor computer code [9]]. As the number of elements used increases, the discontinuity in the stress decreases across element boundaries, and the approximation of equilibrium improves.

Finally, in Figure 3–35, we show the convergence of axial stress at the fixed end $(x = L)$ as the number of elements increases.

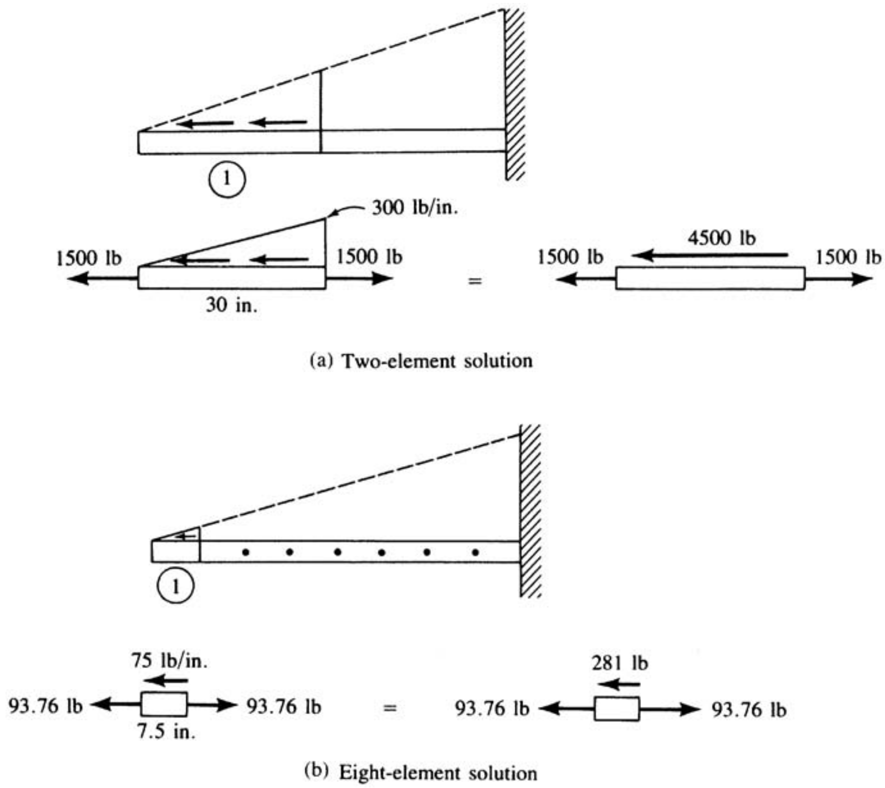


Figure 3–34 Free-body diagram of element 1 in both two- and eight-element models, showing that equilibrium is not satisfied

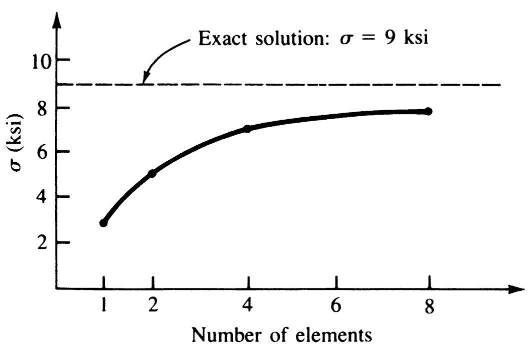


Figure 3–35 Axial stress at fixed end as number of elements increases

However, if we formulate the problem in a customary general way, as described in detail in Chapter 4 for beams subjected to distributed loading, we can obtain the exact stress distribution with any of the models used. That is, letting $\hat{\underline{f}} = \hat{\underline{k}}\hat{\underline{d}} - \hat{\underline{f}}_0$, where $\hat{\underline{f}}_0$ is the initial nodal replacement force system of the distributed load on each element, we subtract the initial replacement force system from the $\hat{\underline{k}}\hat{\underline{d}}$ result. This yields the nodal forces in each element. For example, considering element 1 of the two-element model, we have [see also Eqs. (3.10.33) and (3.10.41)]

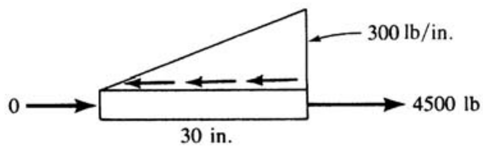
$$\hat{\underline{f}}_0 = \begin{Bmatrix} -1500 \text{ lb} \\ -3000 \text{ lb} \end{Bmatrix}$$

Using $\hat{\underline{f}} = \hat{\underline{k}}\hat{\underline{d}} - \hat{\underline{f}}_0$, we obtain

$$\hat{\underline{f}} = \frac{2(30 \times 10^6)}{(30 \text{ in.})} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -0.006 \text{ in.} \\ -0.00525 \text{ in.} \end{Bmatrix} - \begin{Bmatrix} -1500 \text{ lb} \\ -3000 \text{ lb} \end{Bmatrix}$$

$$= \begin{Bmatrix} -1500 + 1500 \\ 1500 + 3000 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4500 \end{Bmatrix}$$

as the actual nodal forces. Drawing a free-body diagram of element 1, we have



$$\sum F_x = 0: \quad -\tfrac{1}{2}(300 \text{ lb/in.})(30 \text{ in.}) + 4500 \text{ lb} = 0$$

For other kinds of elements (other than beams), this adjustment is ignored in practice. The adjustment is less important for plane and solid elements than for beams. Also, these adjustments are more difficult to formulate for an element of general shape.

3.12 Galerkin's Residual Method and Its Use to Derive the One-Dimensional Bar Element Equations

General Formulation

We developed the bar finite element equations by the direct method in Section 3.1 and by the potential energy method (one of a number of variational methods) in Section 3.10. In fields other than structural/solid mechanics, it is quite probable that a variational principle, analogous to the principle of minimum potential energy, for instance, may not be known or even exist. In some flow problems in fluid mechanics and in mass transport problems (Chapter 13), we often have only the differential equation and boundary conditions available. However, the finite element method can still be applied.