For this project, you are going to study pursuit curves. **Please submit your source code and a Word document or PDF containing the required plots.**

1. [150 points] A pursuit curve is a curve defined by the position of a pursuer, $\boldsymbol{P}$, that is pursuing a target, $\boldsymbol{T}$, that moves in time along a known curve where the motion of $\boldsymbol{P}$ is always directed towards $\boldsymbol{T}$.

   Let the target's position be defined by a given two-dimensional parametric function in time, $\boldsymbol{T}: \mathbb{R} \to \mathbb{R}^2$, and denote the pursuer's position by $\boldsymbol{P}(t) = [x(t), y(t)]^T$. Since the pursuer's motion is always directed towards the target, the tangent line of the pursuit curve is equal to the vector between the target and the pursuer:

$$\frac{\boldsymbol{P}'}{\|\boldsymbol{P}'\|} = \frac{\boldsymbol{T}-\boldsymbol{P}}{\|\boldsymbol{T}-\boldsymbol{P}\|}.$$

   For simplicity, we will assume that the pursuer's speed is proportional to the target's speed for all time: $\|\boldsymbol{P}'\| = k\|\boldsymbol{T}'\|$. Substituting this into the previous equations, we have

$$\boldsymbol{P}' = \frac{k\|\boldsymbol{T}'\|}{\|\boldsymbol{T}-\boldsymbol{P}\|}(\boldsymbol{T}-\boldsymbol{P}).$$

   (a) [20 points] Write a function to compute a pursuit curve given at least the following inputs:
   - Target's position as a function of time, $\boldsymbol{T}(t)$.
   - Speed proportionality constant, k.
   - Pursuer's initial position, $\boldsymbol{P}(0)$.
   - End time for the simulation, t_{end}.
   - (optional) Target's speed function, $\|\boldsymbol{T}'\|$. You can pass this as an input or compute this numerically from $\boldsymbol{T}(t)$.

   The function should provide at least the following outputs:
   - A vector of length N containing the time values where the pursuer's position is calculated.
   - A 2D array of size $(2, N)$ containing the pursuer's position at each value within the time vector.

   (b) [10 points] Write a function that plots the pursuit curve. The plot should include the pursuit curve, the target curve, the start and end positions of both the target and the pursuer, and the appropriate axes. Include in the plot a proper title and appropriately labeled axes.

   (c) [10 points] Write a function that plots the distance between the target and the pursuer as a function of time. Include in the plot an appropriate title and appropriately labelled axes.

   (d) [50 points] Using the above functions, compute and plot the pursuit curves and the distance between the target and pursuer for the following parameters sets:
   1. $k = 1.00$, $\boldsymbol{P}(0) = [1, 0]^T$, and $\boldsymbol{T}(t) = [0, t]^T$ with $t_{\text{end}} = 1$.
   2. $k = 2.00$, $\boldsymbol{P}(0) = [1, 0]^T$, and $\boldsymbol{T}(t) = [0, t]^T$ with $t_{\text{end}} = 2/3$.
   3. $k = 1.00$, $\boldsymbol{P}(0) = [0, 0]^T$, and $\boldsymbol{T}(t) = [\cos(t), \sin(t)]^T$ with $t_{\text{end}} = 2\pi$.
   4. $k = 1.22$, $\boldsymbol{P}(0) = [0, 0]^T$, and $\boldsymbol{T}(t) = [\cos(t), \sin(t)]^T$ with $t_{\text{end}} = \pi/2$.
   5. $k = 0.50$, $\boldsymbol{P}(0) = [0, 0]^T$, and

$$\boldsymbol{T}(t) = \begin{bmatrix} (R+r)\cos(t) - d\cos\left(\frac{R+r}{r}t\right) \\ (R+r)\sin(t) - d\sin\left(\frac{R+r}{r}t\right) \end{bmatrix}$$

      with $R = 7$, $r = 6$, $d = 9$, and $t_{\text{end}} = 4\pi r$.

   You must use compute the solution at a sufficient number of time values to generate smooth curves.

   (e) [20 points] Write a function that approximates the critical speed proportionality constant, k_{crit}, such that the pursuer catches the target at a given time $t = t_{\text{catch}}$. Define catching the target as the distance between the pursuer and the target has become less than 10^{-4}.

   (f) [40 points] Compute the critical speed proportionality constant and plot the pursuit and target curves for the catch times: $t_{\text{catch}} = \pi/4, \pi/2, \pi$, and 2π. Assume for all four cases that $\boldsymbol{P}(0) = [0, 0]^T$, and $\boldsymbol{T}(t) = [\cos(t), \sin(t)]^T$. Run your simulations until the target is caught.

Analytical Solution:

We will restrict our target's motion to a vertical line to generate an analytical solution for testing purposes. Letting

$$\boldsymbol{T}(t) = \begin{bmatrix} 0 \\ t \end{bmatrix} \quad \text{and} \quad \boldsymbol{P}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \tag{1}$$

our system of differential equations becomes

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{k}{\sqrt{x^2 + (t-y)^2}} \begin{bmatrix} -x \\ t-y \end{bmatrix} \tag{2}$$

since $\|\boldsymbol{T}'(t)\| = 1$. We now dot the equation with $\boldsymbol{P}'$:

$$\|\boldsymbol{P}'\|^2 = \frac{k}{\sqrt{x^2 + (t-y)^2}} \begin{bmatrix} x' \\ y' \end{bmatrix} \cdot \begin{bmatrix} -x \\ t-y \end{bmatrix}. \tag{3}$$

Since $\|\boldsymbol{P}'\| = k\|\boldsymbol{T}'\| = k$, after rearranging, we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} \cdot \begin{bmatrix} -x \\ t-y \end{bmatrix} = k\sqrt{x^2 + (t-y)^2}. \tag{4}$$

We expand the dot product and then square both sides:

$$[y'(t-y) - x'x]^2 = k^2\left(x^2 + (t-y)^2\right) \tag{5}$$

After expanding, we have

$$x^2\left[(x')^2 - k^2\right] - 2x(t-y)x'y' + (t-y)^2\left[(y')^2 - k^2\right] = 0 \tag{6}$$

Note that $(x')^2 + (y')^2 = k^2$ from $\|\boldsymbol{P}'\|^2 = k^2\|\boldsymbol{T}'\|^2$ so we can simplify our equation to obtain

$$x^2(y')^2 + 2x(t-y)x'y' + (t-y)^2(x')^2 = 0 \tag{7}$$

This is a complete square:

$$[xy' + (t-y)x']^2 = 0 \tag{8}$$

so we now have the differential equation

$$xy' + (t-y)x' = 0 \tag{9}$$

Divide the equation by x' and denote $dy/dx = y'/x'$ to obtain

$$x\frac{dy}{dx} - y = t \tag{10}$$

We need to eliminate t from the equation. We note that the distance traveled by $\boldsymbol{P}$ equals the time elapsed multiplied by the pursuer's speed. For this problem, that is $t\|\boldsymbol{P}'\| = tk\|\boldsymbol{T}'\| = tk$. Since the distance traveled is the arc length, we have

$$s = \int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = kt \tag{11}$$

Solving this for t and substituting into our differential equation, we have

$$x\frac{dy}{dx} - y = \frac{1}{k}\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx. \tag{12}$$

Taking the derivative with respect to x, we have the second-order equation

$$x\frac{d^2y}{dx^2} - \frac{1}{k}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0 \tag{13}$$

Let $w = dy/dx$ to obtain the first-order equation

$$x\frac{dw}{dx} - \frac{1}{k}\sqrt{1 + w^2} = 0, \tag{14}$$

which is separable:

$$\int \frac{dw}{\sqrt{1 + w^2}} = \int \frac{dx}{kx}. \tag{15}$$

The implicit solution is

$$\sinh^{-1}(w) = \frac{1}{k}\ln(x) + C \tag{16}$$

which, letting $a = k^{-1}$, simplifies to

$$w(x) = \sinh(\ln(x^a) + C) = \frac{1}{2}\left[e^{\ln(x^a)+C} - e^{-(\ln(x^a)+C)}\right] = \frac{1}{2}\left[Cx^a - \frac{1}{Cx^a}\right] \tag{17}$$

Integrating $dy/dx = w$, we obtain, for $a \neq 1$,

$$y(x) = C_1 + C_2\frac{x^{a+1}}{a+1} + \frac{1}{4C_2}\frac{x^{1-a}}{a-1} \tag{18}$$

and, for $a = 1$,

$$y(x) = C_1 + C_2x^2 - \frac{1}{8C_2}\ln(x) \tag{19}$$

Assuming the initial condition $\boldsymbol{P}(0) = [x_0, y_0]^T$, we can form two initial conditions for our $y = y(x)$:

$$y(x_0) = x_0 \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{x=x_0} = \frac{y_0}{x_0}. \tag{20}$$

Enforcing the conditions yield, for $a \neq 1$,

$$y(x) = y_0 - \frac{1}{2}\left[\frac{y_0 + r_0}{1+a}\left(1 - \left(\frac{x}{x_0}\right)^{1+a}\right) - \frac{r_0 - y_0}{1-a}\left(1 - \left(\frac{x}{x_0}\right)^{1-a}\right)\right] \tag{21}$$

and, for $a = 1$,

$$y(x) = \frac{1}{4}\left[3y_0 - r_0 + (y_0 + r_0)\left(\frac{x}{x_0}\right)^2 - 2(r_0 - y_0)\ln\left(\frac{x}{x_0}\right)\right]. \tag{22}$$

To determine the parametric functions $x(t)$ and $y(t)$, we would have to solve the nonlinear equation

$$t = x\frac{dy}{dx} - y \tag{23}$$

for $x(t)$ and the substitute to obtain $y(x(t))$. This would be an alternative to solving the system of ODEs but would only apply to the case where $\boldsymbol{T}(t) = [0, t]^T$.

For $k > 1$, which implies that $a < 1$, the pursuer moves faster than the target so the pursuer can catch the target. For an assumed target trajectory, that will be when $x = 0$, so the pursuer will catch the target at the point

$$(0, y(0)) = \left(0, \frac{a}{1-a^2}\right) = \left(0, \frac{k}{k^2-1}\right) \tag{24}$$

Since $y = t$ for the target, the catch time would be $t_{\text{catch}} = k/\left(k^2 - 1\right)$.

When $k = 1$, we can determine the limiting distance between the target and the pursuer as $t \to \infty$. Recall that the time kt is the arc length of the solution, so

$$\frac{dt}{dx} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = -\sqrt{1 + \frac{1}{4}\left(\frac{x}{r_0 - y_0} - \frac{r_0 - y_0}{x}\right)^2}. \tag{25}$$

Note that we negate the derivative since t increases as x decreases. Integrating in x yields

$$t = C_3 + \frac{1}{4}\left[\frac{x^2}{r_0 - y_0} - 2\left(r_0 - y_0\right)\ln\left(\frac{x}{y_0 - r_0}\right)\right] \tag{26}$$

We fix the constant by noting that $x = x_0$ at $t = 0$ and using $x_0^2 = r_0^2 - y_0^2$ to obtain

$$t = \frac{1}{4}\left[\frac{x_0^2 - x^2}{r_0 - y_0} - 2\left(r_0 - y_0\right)\ln\left(\frac{x}{x_0}\right)\right] \tag{27}$$

We then compute the limit

$$\lim_{t\to\infty} \left\|\boldsymbol{T}(t) - \boldsymbol{P}(t)\right\|. \tag{28}$$

For our example, we have

$$\lim_{x\to 0^+}\left(t - y(x)\right) = \lim_{x\to 0^+} \frac{r_0 - y_0}{2}\left[1 - \left(\frac{x}{r_0 - y_0}\right)^2\right] - \frac{r_0 - y_0}{2} \tag{29}$$

For the first parameter set in Part (d), the pursuit curve and the distance between the target and the pursuer are shown below. The limiting distance is $\lim_{t\to\infty}\left\|\boldsymbol{T}(t) - \boldsymbol{P}(t)\right\| = 1/2$ as shown.