9

Quantifying Chaos

Let chaos storm, Let cloud shapes swarm! I wait for form. Robert Frost, *Pertinax*

9.1 Introduction

How chaotic is a system's chaotic behavior? In this chapter we shall discuss several ways to give a quantitative answer to that question. Before we get immersed in the details of these answers, we should ask ***why*** we might want to quantify chaos. One answer lies in a desire to be able to specify quantitatively whether or not a system's apparently erratic behavior is indeed chaotic. As we have seen chaotic behavior generates a kind of randomness and loss of information about initial conditions, which might explain complex behavior (or at least some of the complex behavior) in real systems. We would like to have some definitive, quantitative way of recognizing chaos and sorting out "true" chaos from just noisy behavior or erratic behavior due to complexity (that is, due to a large number of degrees of freedom). Second, as we shall see in the next chapter, some of these quantifiers can give us an estimate of the number of (active) degrees of freedom for the system. A third reason for quantifying chaotic behavior is that we might anticipate, based on our experience with the universality of the scenarios connecting regular behavior to chaotic behavior, that there are analogous universal features, perhaps both qualitative and quantitative, that describe a system's behavior and changes of its behavior within its chaotic regime as parameters of the system are changed. We will see that indeed some such universal features have been discovered and that they seem to describe accurately the behavior of actual systems. Finally, (although this is rarely possible today), we would hope to be able to correlate changes in the quantifiers of chaotic behavior with changes in the physical behavior of a system. For example, is there some quantifier whose changes are linked to the onset of fibrillation in heartbeats or the beginnings of turbulence in a fluid or noisy behavior in a semiconductor circuit?

In addition to calculating values for particular quantifiers for chaotic systems, we need to be able to estimate uncertainties associated with those quantifiers. Without those uncertainties, it is impossible to make meaningful comparisons between experimentally measured and theoretically calculated values or to compare results from different experiments. We will suggest several ways of estimating these uncertainties in our discussion.

To summarize, here are some reasons for quantifying chaotic behavior:

1. The quantifiers may help distinguish chaotic behavior from noisy behavior.

2. The quantifiers may help us determine how many variables are needed to model the dynamics of the system.

3. The quantifiers may help us sort systems into universality classes.

4. Changes in the quantifiers may be linked to important changes in the dynamical behavior of the system.

9.2 Time-Series of Dynamical Variables

The key theoretical tool used for quantifying chaotic behavior is the notion of a time-series of data for the system. We met up with this idea in Chapter 1 in the form of a stroboscopic portrait of the current in the semiconductor diode circuit and later in the more general form of Poincaré sections in state space. In this chapter we will focus on using a time-sequence of values of a single system variable, say $x(t)$, to determine the quantitative measures of the system's (possibly) chaotic behavior. We will assume that we have recorded a sequence of values $x(t_0)$, $x(t_1)$, $x(t_2)$, ... with $t_0 < t_1 < t_2$, and so on, as illustrated in Fig. 9.1. This could be a series of time-sampled values of some variable, where the time values are fairly close together, or it could be a series of Poincaré section values for some variable at fairly widely separated time values.

It is not obvious that such a set of sampled values of just one variable should be sufficient to capture the features we want to describe. In fact, as we shall argue in the next chapter, if the sampling is carried out at appropriate time intervals (which we shall need to specify) and if the sequence is used cleverly, then we can indeed "reconstruct" the essential features of the dynamics in state space. We will show in Chapter 10 that we can often determine the number of state variables needed to specify the state of the system from the time record of just one variable.

Of course, we need to say what we mean by essential. Sampled values of one variable will clearly not (or, in general, cannot) tell us what the other variables are doing (unless we happen to have a complete theory for the system). If we limit our goals, however, to recognizing bifurcations in the system's behavior and determining if the behavior is chaotic and if so, how chaotic, then it turns out that this single variable sequence is sufficient (with some qualifications, of course).

One further comment on measuring a single variable is in order. In almost all measurements, our instruments measure the dynamical variables indirectly. For example, if we are interested in temperature, we may actually measure the voltage produced by a thermocouple in contact with our system. We generally assume that our "measurement function" provides a fairly straightforward representation of the

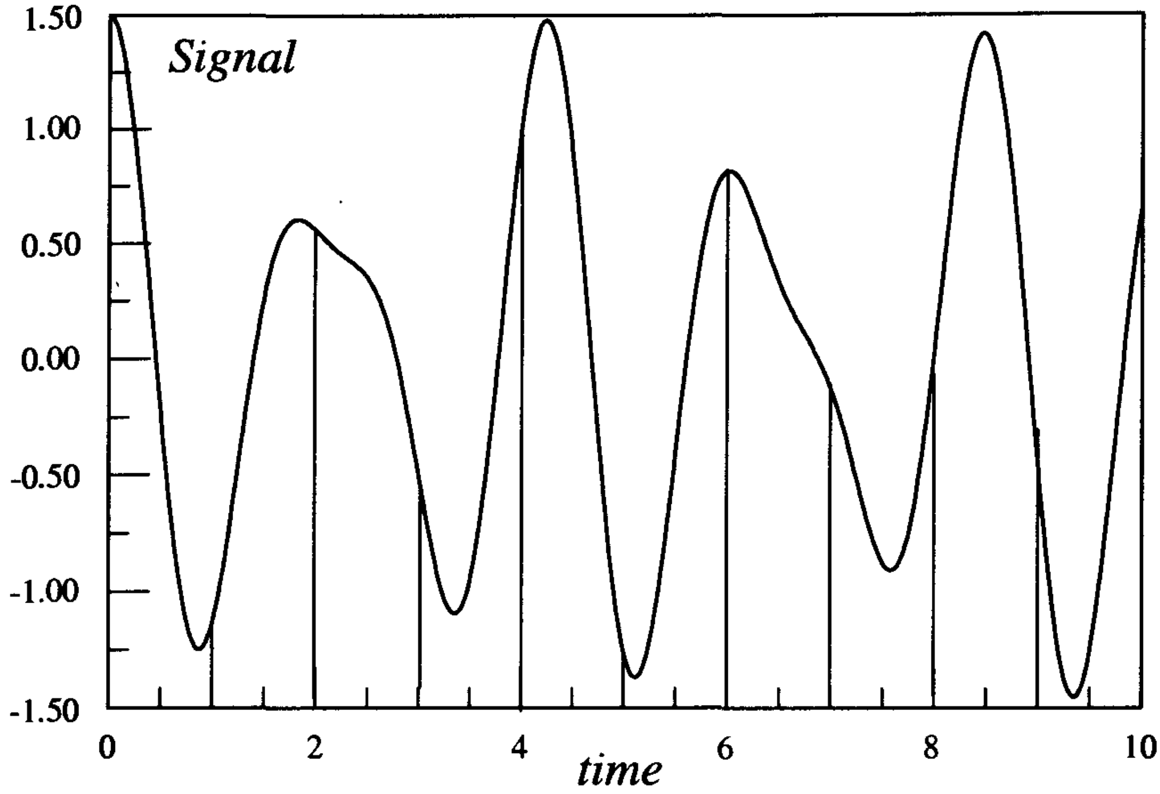

Fig. 9.1. A sketch of the sampling of a dynamical variable. The recorded values form the time series for data analysis. Here the sampling is done at t = 0, 1, 2, . . . and so on.

actual quantity we want to monitor. Strictly speaking, however, we are monitoring the dynamics of our measurement function, not the system directly.

It might be tempting to base our analysis of the system's behavior on continuous time trajectories, given symbolically as $\vec{x}(t)$, where the vector quantity represents a complete set of dynamical variables for the system. (A complete set is the minimum number of variables needed to specify uniquely the state of the system.) In this kind of analysis, the value of $\vec{x}$ is available for any value of the time parameter. However, real experiments always involve discrete time sampling of the variables, and numerical calculations, which we must use for most nonlinear systems, always have discrete time steps. Since both real experiments and actual computer calculations always give the variable values in discrete time steps, we make a virtue of necessity and base our entire discussion on these discrete time sequences.

The problem of choosing the appropriate time between samples (that is, choosing $t_1 - t_0$, $t_2 - t_1$, etc.) is a delicate one. If an infinite amount of noise-free data is available, then almost any set of time intervals will do. However, for more realistic situations, with a finite amount of data contaminated by some noise, we must proceed very cautiously. In the next chapter, we shall develop some "rules of thumb" for selecting time sample intervals and other features of the data. The reader who wants to undertake this kind of analysis for her or his data should consult Chapter 10 and the references at the end of this chapter for more details on